Robust capacity assignment solutions for telecommunications networks with uncertain demands

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Abstract

We consider the capacity planning of telecommunications networks with linear investment costs and uncertain future traffic demands. Transmission capacities must be large enough to meet, with a high quality of service, the range of possible demands, after adequate routings of the traffic on the created network. We use the robust optimization methodology to balance the need for a given quality of service with the cost of investment. Our model assumes that the traffic for each individual demand fluctuates in an interval around a nominal value. We use a refined version of affine decision rules based on a concept of demand proximity to model the routings as affine functions of the demand realizations. We then give a probabilistic analysis assuming the random variables follow a triangular distribution. Finally, we perform numerical experiments on network instances from SNDlib and measure the quality of the solutions by simulation.

Keywords. Capacity assignment problem, Telecommunication networks, Robust optimization.

1 Introduction

Two conflicting criteria guide operators when they design or reinforce telecommunications networks: the cost of investment and the quality of service (QoS) (expressed here as a level of rejected requests). On an existing network, failure to achieve full service is caused by an excessive traffic load in conjunction with insufficient installed capacity. When facing uncertain traffic demands, the network designers may be tempted to over-estimate the traffic demand in order to achieve the required QoS. This is likely to result in oversized networks entailing unnecessarily large investment cost. Our goal is to show that an approach based on robust optimization is able to achieve the required QoS at a reduced cost.

The need for new capacity occurs when the telecommunications operator offers new services or has to cope with growing traffic. For strategic and operational reasons those investments must be planned well in advance and decisions must rely on traffic forecasts,

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which are inherently uncertain. Analysis of real-life data have revealed large errors between marketing forecasts and the actual future traffic in the network. Even for a one year horizon, the difference can easily be larger than 10%. It is thus often the case that when the planned network becomes operational, it turns out to be inadequately sized. Capacity adjustments must be made, at possibly much higher cost.

The standard formulation of the capacity planning problem for telecommunications networks is a deterministic integer multicommodity flow problem with modular investment costs and possibly some costs associated with the traffic routing (e.g., travel times). Additional constraints to ensure traffic routing diversification may also be included. In a short-term environment, deterministic models for capacity planning are adequate, but this is not so in long-term analyses. Demand forecasts for future years ahead are usually point estimates prone to significant errors. It is thus difficult to make meaningful studies which incorporate the combinatorial options induced by modular investments and the risk of failure to serve a demand that can severely depart from the base forecast. Therefore, long-term planners tend to focus on getting a schematic view of the future network architecture and a sound estimate of the global investment budget. The main issue at stake is the robustness of the projected network architecture with respect to the future demands. In contrast, the modular structure of the invested capacity is only moderately relevant. As a consequence, the literature on capacity investment in telecommunications network with uncertain demand generally assumes that the investment costs are linear. We follow this route in the present paper.

When demands are uncertain, a two-stage stochastic programming formulation is quite natural [12, 23]. In the first stage, capacities are allocated as deterministic variables; routings to meet the uncertain demands are set in a second stage. Routings are thus stochastic by construction. This problem is a special case of more general formulations of network design in which the demands and possibly the travel times with a single commodity and multiple sources and sinks or with multiple commodities with single source and a single sink for each commodity.

A first issue concerns the modeling of uncertain demands. Some contributions use a scenario-based description with the condition that for each demand scenario there is a feasible flow that meets the demand and the capacity constraint on the installed capacities [12, 19, 21, 22, 23]. A solution that satisfies this requirement is claimed to be robust. In view of the linearity of all constraints, a solution that is robust with respect to all scenarios is also robust with respect to the convex hull of the scenarios. Therefore the demand uncertainty can be described via a polyhedral uncertainty set. This idea has been specifically exploited in [5, 11], where the set of traffic events is defined through inequalities rather than from a set of discrete scenarios. More general convex-compact uncertainty sets are considered in [8] (see also [9, 18]). Those sets, just like the scenario-based uncertainty sets, do not cover the whole realm of possible demand realizations. Those sets can be made more or less large by using a homothetic factor usually named the *immunization factor*. The larger the immunization factor, the more demand realizations are taken into account. A solution is claimed robust if for each demand in the uncertainty set there is a routing that meets the demands and is compatible with the installed capacity. A third alternative is to describe uncertainty via a probability distribution and use a chance-constrained programming formulation [14]. It is known that the feasible set with probabilistic constraints is in general not convex, and possibly disconnected, a fact that also leads to considerable numerical difficulties.

The second issue concerns the handling of the routing decisions that are recourse decisions adjustable to meet the observed demands. As pointed out in [6, 7] fully adjustable solutions are liable to make the computation of robust solutions intractable, a fact that is recognized in [16, 18, 22]. To get around intractability, reference [7] proposes a revival of the concept of Affine Decision Rules (ADR). In that framework, recourse decisions are formulated as affine functions of the observed demand uncertainties (see [20]). In our problem of interest when the number of demands is large this option can lead to a significant increase in the number of variables. In the present paper, we use a refined version of ADR based on a concept of demand proximity to overcome this difficulty.

The current paper builds on earlier work [20] of two of the present authors. The deterministic version of the model is a path-flow formulation. A set of admissible paths is proposed as part of the data. The number of paths used by the solution may be restricted to a fixed cardinality subset of the admissible paths. Individual demands are modeled as independent[1] uncertain variables. Each one varies around a nominal value on a symmetric interval. First stage decisions deal with the investment for capacity, while the recourse decisions are the actual routings on the network to meet the observed demands. Recourse decisions are modeled as ADR[2]. The uncertainty sets are polyhedral. We chose not to use ellipsoidal sets in order to remain in the realm of linear programming and make it easier to handle first stage decision variables with integrality restrictions. We impose that the ADR's be such that the demand requirement is satisfied for all demand scenarios within the uncertainty set. However ADR's spread the uncertainty into the capacity constraints and the flow nonnegativity constraints. Those two sets of constraints are robustified. To quantify the immunization factor in the equivalent robust counterpart, we perform a probabilistic analysis under the assumption that each demand is distributed according to a triangular symmetric distribution. A weaker assumption could be made, that would give very similar, though weaker, results. For a detailed study, we refer to [6, 8, 9]. By construction, a robust solution tends to be conservative for three main reasons: i) ADRs are suboptimal; ii) the immunization factor that guarantees a certain probability of satisfaction is over-estimated by the theoretical analysis based on probabilistic inequalities; iii) the indicators of constraint violation are not independent random variables from constraint to constraint. Consequently, the practical solution must be evaluated empirically, using simulations.

In [18] the authors consider a related but different problem of capacity expansion involving the operating cost. They use robust optimization with simple box uncertainty. Under this assumption, the worst case occurs when each individual demand is at its peak value: there is no need to resort to ADRs to handle the adjustable routing variables.

The paper is organized as follows. In Section 2 we present the basic capacity assignment problem. In Section 3 , we propose a model for the uncertainty on the demands and give the formulation of affine decision rules for the recourse. Section 4 formulates the robust counterpart of an uncertain constraint and collects a few known results in robust optimization. It also provides a probabilistic analysis with the triangular distribution. In Section 5, we model the capacity planning problem with robust constraints. Section 6 is devoted to numerical results on test problems. The QoS achieved by the robust solutions is observed by simulation on a Monte-Carlo sample of demand scenarios, assuming a triangular distribution. Section 7 is a conclusion with some hints for further research.

[1]This choice is dictated by the situation that is prevalent in practice. More sophisticated models, involving, for instance, factors and correlations, can equally be handled by robust optimization. However, it turns out that, in our problem of interest of new telecommunications services, the influence factor matrices and the correlation matrices are generally not available or not even computable.

[2]Or affinely adjustable variables in the parlance of [7].

2 Capacity assignment problem

Let $\mathcal{G}(\mathcal{N}, \mathcal{A})$ be a directed graph where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of arcs. We denote by $\mathcal{K}$ the set of demands, characterized by origin-destination (OD) pairs. Let d_k be the traffic amount for demand $k \in \mathcal{K}$, and let $\mathcal{I}_k$ denote the set of available paths that can be used to route the demand k. This set $\mathcal{I}_k$ may contain all paths between the origin and the destination of demand k. However, it is often desirable, or even required, to restrict the number of potential paths for a given demand: indeed, not all the paths are acceptable by network managers; furthermore, from a computational viewpoint, this restriction will be helpful.

The path-flow formulation of the capacity assignment problem with linear assignment cost is:

$$\min_{f\geq 0, c\geq 0} \quad \sum_{a\in\mathcal{A}} l_a c_a \tag{1a}$$

$$\text{s.t.} \quad \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}_k} \pi^a_{ik} f_{ik} \leq c_a, \quad a \in \mathcal{A} \tag{1b}$$

$$\sum_{i\in\mathcal{I}_k} f_{ik} = d_k, \quad k \in \mathcal{K}. \tag{1c}$$

In this formulation l_a is the cost of installing a unit capacity on arc a. The vector π_{ik} has component $\pi^a_{ik} = 1$ if path $i \in \mathcal{I}_k$ includes arc a, and 0 otherwise. The flow for the traffic demand k on the path i is represented by the variable f_{ik}. Finally, variable c_a is the capacity to be installed on the arc a. The above formulation can easily be extended to handle modular restriction on the investment for capacities. For instance, capacities can be restricted to multiples of a base capacity. As discussed in the introduction, this is not too relevant in long-term planning. Furthermore, deterministic problems with modular capacities are already very difficult [2, 3].

In this problem, the number of admissible paths is restricted and is given in explicit form. Practitioners usually add the further restriction that only a small number of those admissible paths can be effectively used in the proposed solution. Let m_k be this number for the demand k. To deal with this restriction, we add binary variables and reformulate the original problem as the mixed integer linear problem (MIP)

$$\min_{f,c,y} \quad \sum_{a\in\mathcal{A}} l_a c_a \tag{2a}$$

$$\text{s.t.} \quad \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}_k} \pi^a_{ik} f_{ik} \leq c_a, \quad a \in \mathcal{A} \tag{2b}$$

$$\sum_{i\in\mathcal{I}_k} f_{ik} = d_k, \quad k \in \mathcal{K} \tag{2c}$$

$$\sum_{i\in\mathcal{I}_k} y_{ik} \leq m_k, \quad k \in \mathcal{K} \tag{2d}$$

$$f_{ik} \leq M_k y_{ik}, \quad k \in \mathcal{K},\ i \in \mathcal{I}_k \tag{2e}$$

$$f \geq 0, c \geq 0, \tag{2f}$$

$$y_{ik} \in \{0, 1\}, \quad k \in \mathcal{K},\ i \in \mathcal{I}_k. \tag{2g}$$

The binary variables y indicate whether a path is opened or not. The parameter M_k is an upper bound on the flow f_{ik}. One can take $M_k = d_k$ or a tighter bound if available. Constraints (2d) limit the number of active paths.

One of the major problems of this model lies in the values of traffic demand d to be used. In practice, network operators use statistical models, together with market surveys, to forecast the evolution of traffic. However, experience shows that forecasts are always wrong, and often far from the observed reality. Traffic measurements on a backbone network of France Telecom, compared with the amounts forecasted one year before, have revealed gaps up to 25% on the global amount of traffic in the network (i.e., $\sum_{k\in\mathcal{K}} d_k$), depending on the year. Note that the error may be positive, as well as negative, around the expected value. If no probabilistic assumption is performed, this error of 25% should be reported on all demands $\{d_k\}_{k\in\mathcal{K}}$.

Under some circumstances, the error on individual demands may be far larger than that. Indeed, given the successive errors performed year after year, we estimate the average error and its standard deviation. In our data, we have observed a standard deviation of 17% for the error on $\sum_{k\in\mathcal{K}} d_k$. For the sake of simplicity, let us assume that the error process is Gaussian, and that demands $\{d_k\}_{k\in\mathcal{K}}$ are independent Gaussian random variables, with standard deviations $\sigma_k = \tau \cdot \mathbb{E}[d_k]$, for some $\tau > 0$. Then we have: $\left(17\% \cdot \mathbb{E}\left[\sum_{k\in\mathcal{K}} d_k\right]\right)^2 = \sum_{k\in\mathcal{K}} \left(\tau \cdot \mathbb{E}[d_k]\right)^2$, which implies:

$$\tau = 17\% \cdot \frac{\sum_{k\in\mathcal{K}} \mathbb{E}[d_k]}{\sqrt{\sum_{k\in\mathcal{K}} \mathbb{E}[d_k]^2}}.$$

The relative error on each individual demand d_k may thus be far larger than the relative error on the global traffic amount (τ could reach $17\% \cdot \sqrt{|\mathcal{K}|}$, if all $\mathbb{E}[d_k]$ are equal).

Finally, the above quantitative analysis is based on forecasts performed one year in advance. Of course, the uncertainty is much larger if forecasts are performed earlier: even only two years in advance, we observed errors more than 40% on the global traffic amount. Notwithstanding, network operators usually perform network plans several years in advance.

The models developed in this paper aim at tackling this uncertainty issue.

3 Uncertainty and affine decision rules (ADR)

The capacity investment problem is essentially a two-stage problem with recourse. In the first stage, the capacity investment is selected. In the second stage, the decision concerns traffic routing to meet the observed demands. The sequence is thus: invest in capacities, observe the demands and route the traffic. The important feature is the flexibility brought in by the recourse nature of the routing process. We propose to capture this feature via *decision rules*. To do so we must formalize first the demand process and next the decision rule.

In real life, defining a probability distribution for the demand may be difficult. It is far more natural and easier for practitioners to provide a range around a nominal value for each individual demand. This information can be captured by the equation:

$$d_k = \bar{d}_k + \hat{d}_k \xi_k, \qquad (3)$$

where ξ_k represents a random factor taking values in [-1,1], $\bar{d}_k$ is the average demand and $\hat{d}_k$ is the demand dispersion. The demand range is thus $[\bar{d}_k - \hat{d}_k, \bar{d}_k + \hat{d}_k]$.

Remark 1 *More complex models could be used, in particular to capture correlation. In that case, one would adopt a model like*

$$d = \bar{d} + \hat{D}\xi$$

where $\bar{d} \in \mathbb{R}^{|\mathcal{K}|}$ is the reference demand (e.g., the estimated average, or the estimated mode), $\xi \in \mathbb{R}^p$ is the random factor of perturbation and $\hat{D}$ is a $|\mathcal{K}| \times p$ factor matrix. If $E(\xi) = 0$ and $E(\xi\xi^T) = \rho^2 I$, then $\bar{d} = E(d)$ and $E(dd^T) = \rho^2 DD^T$. The matrix ρD appears to be a square root of the order 2 moment matrix[3] of d. Unfortunately, moment data are not available in practice, and very little information can be obtained on the possible realizations of the demands. For these reasons, we stick with the simpler Model 3.

Given the demand model, we can now define a decision rule as a function from the space of demand realizations to the space of recourse decision variables. The concept of decision rule captures the fact that the recourse decisions need not be fixed before the demand is realized and can be adjusted to fit the observed demand. An ADR is just a special case of this general setting [6, 7]. Linearity makes them computationally tractable, but possibly sub-optimal. In our case, the decision rule will apply to the routing decisions, that is to the flows on the admissible paths. An ADR will make each individual flow an affine function of all demand uncertainties:

$$f_{ik} = \alpha_{0ik} + \sum_{k' \in \mathcal{K}} \alpha_{k'ik} \xi_{k'}, \ \forall k \in \mathcal{K}, \ i \in \mathcal{I}_k. \tag{4}$$

In this representation, the new decision variables are the coefficients $\alpha_{0ik} \in \mathbb{R}$ and $\alpha_{k'ik} \in \mathbb{R}$.

In view of the large number of demands, i.e., the large cardinality of $\mathcal{K}$, the number of variables is large and the robust model to be developed in the later section may become excessively large. Though well founded in theory, this approach may not lead to significantly better results than the simpler one in which the decision rule depends on a lesser number of factors. In [20], the authors restricted the decision rule for the OD pair k to be a function of only two factors: the traffic amount d_k and the sum of all other demands. In the present paper we intend to use a more refined version in which for each OD pair k, the set of other demands are partitioned on the basis of a proximity concept. The OD pairs $k' \in \mathcal{K} \setminus \{k\}$ that are deemed "close" to k are gathered in the set V_k. The other demands are collected into $R_k = \{k' \in \mathcal{K} | k' \neq k, k' \notin V_k\}$. With these notations we define the decision rule as the affine function:

$$f_{ik} = \alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik} \sum_{k' \in V_k} \xi_{k'} + \alpha_{3ik} \sum_{k' \in R_k} \xi_{k'}, \ \forall k \in \mathcal{K}, \ i \in \mathcal{I}_k \tag{5}$$

With this rule, the routing for each demand is defined by exactly four coefficients α. The key issue is of course the type of proximity to be used. We tried different types, but we eventually retained the following one.

Definition 1 *The traffic demand k' is close to the demand k if their respective shortest paths have at least one arc in common. Shortest path for demand k corresponds to the path among the set $\mathcal{I}_k$ with the smallest investment cost.*

This proves to be sufficient in practice.

Denoting $\xi_k^v = \sum_{k' \in V_k} \xi_{k'}$ and $\xi_k^r = \sum_{k' \in R_k} \xi_{k'}$ we may rewrite (5) in the more compact form:

$$f_{ik} = \alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik}\xi_k^v + \alpha_{3ik}\xi_k^r. \tag{6}$$

[3]The square root is unique if $E(dd^T)$ has full dimension.

Upon replacing the flow variables by their ADR in problem (2) we get, for any single traffic event ξ:

$$\min_{\alpha, c \geq 0, y} \quad \sum_{a \in \mathcal{A}} l_a c_a \tag{7a}$$

$$\text{s.t.} \quad \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}_k} \pi^a_{ik} \left(\alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik}\xi^v_k + \alpha_{3ik}\xi^r_k \right) \leq c_a, \quad a \in \mathcal{A} \tag{7b}$$

$$\sum_{i \in \mathcal{I}_k} (\alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik}\xi^v_k + \alpha_{3ik}\xi^r_k) = \bar{d}_k + \xi_k \hat{d}_k, \quad k \in \mathcal{K} \tag{7c}$$

$$y_{ik} M_k \geq \alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik}\xi^v_k + \alpha_{3ik}\xi^r_k \geq 0, \quad k \in \mathcal{K}, i \in \mathcal{I}_k \tag{7d}$$

$$\sum_{i \in \mathcal{I}_k} y_{ik} \leq m_k, \quad k \in \mathcal{K} \tag{7e}$$

$$y_{ik} \in \{0, 1\}, \quad k \in \mathcal{K}, i \in \mathcal{I}_k. \tag{7f}$$

In this model, the uncertainty creeps into most constraints, either by definition like in (7c), or via the decision rule like in (7b) and (7d). We shall use the robust optimization paradigm to handle uncertainty in those constraints.

4 Robust counterpart of an uncertain constraint

In this section, we consider the single linear constraint with uncertain coefficient

$$\tilde{a}(\xi)^T x \leq b \tag{8}$$

where $x \in \mathbb{R}^n$ is the decision variable, $\xi \in \mathbb{R}^m$ is a random factor and $b \in \mathbb{R}$ is a constant[4] term. Note that all the constraints of formulation (7) can be written in this form. Even though the focus of the paper is on modeling and experiments, we choose to reproduce and adapt some known facts and results about robust optimization and safe tractable convex approximation of chance constraints. We hope it helps the reader to put the study in the proper perspective.

The first basic assumption on the uncertain parameters is that they depend on some random factor ξ in a linear way.

Assumption 1 *The uncertain vector $\tilde{a}$ is written as:*

$$\tilde{a}(\xi) = \bar{a} + P\xi,$$

where $\xi \in [-1, 1]^m$ and P is an $n \times m$ matrix.

The certain vector $\bar{a}$ is usually named the normal factor.

4.1 Robust counterpart

The robust version of the initial uncertain constraint $\tilde{a}(\xi)^T x \leq b$ aims at enforcing feasibility of the uncertain constraint (8), not for *all* possible realizations, but only for a subset named the *uncertainty set*. The *robust counterpart* of the uncertain constraint (8) is

$$\bar{a}^T x + (P^T x)^T \xi \leq b, \text{ for all } \xi \in \Xi, \tag{9}$$

[4] b can be made dependent of the random factor. It suffices to introduce an extra decision variable x_{n+1} with the constraint $x_{n+1} = 1$.

where $\Xi \subset \mathbb{R}^m$ is the uncertainty set. A solution to this constraint is called *robust* with respect to Ξ. If Ξ is a continuous set, the robust counterpart is a semi-infinite constraint, which is potentially numerically intractable. However, for a large class of uncertainty sets, one can give an equivalent form in terms of finitely many tractable constraints.

In this paper we shall use the general uncertainty set

$$\Xi = \{\xi \mid \|\xi\|_\infty \leq 1,\ \|\xi\|_p \leq \kappa_p\} \tag{10}$$

with $p = 1$ or $p = 2$. Parameter κ_p will be defined later. Using standard duality theory for convex programming one can easily prove (see e.g., [4])

Theorem 1 *Let $z \in \mathbb{R}^m$. Then*

$$\max_{\xi \in \mathbb{R}^m} \{z^T\xi \mid \|\xi\|_\infty \leq 1,\ \|\xi\|_p \leq \kappa_p\} = \min_{w \in \mathbb{R}^m} \{\kappa_p \|z^T x - w\|_q + \|w\|_1\},$$

with $q = \infty$ if $p = 1$ and $q = 2$ if $p = 2$.

This result is used to write down the robust counterpart by setting $z = P^T x$ and dispensing with the min operator.

Corollary 1 *The robust equivalent of the robust constraint*

$$\bar{a}^T x + (P^T x)^T \xi \leq b, \text{ for all } \xi \in \Xi = \{\xi \mid \|\xi\|_\infty \leq 1,\ \|\xi\|_p \leq \kappa_p\},$$

is the constraint in $x \in \mathbb{R}^n$ **and** *$w \in \mathbb{R}^m$*

$$\bar{a}^T x + \kappa_p \|P^T x - w\|_q + \|w\|_1 \leq b. \tag{11}$$

The equivalent robust counterpart is a numerically tractable conic constraint for both values $p = 2$ and $p = 1$. In the former case, the constraint is conic quadratic. In the second case, the constraint is amenable to a set of linear constraints, as stated in the next theorem.

Theorem 2 *The conic constraint*

$$\bar{a}^T x + \kappa_1 \|P^T x - w\|_\infty + \|w\|_1 \leq b \tag{12}$$

has the same set of solutions as the system of linear inequalities

$$\bar{a}^T x + \kappa_1 t + e^T w \leq b \tag{13a}$$

$$-(w + te) \leq P^T x \leq w + te \tag{13b}$$

$$w \geq 0,\ t \geq 0 \tag{13c}$$

in the variables $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m_+$, $t \in \mathbb{R}^n_+$. Here $e \in \mathbb{R}^m$ is the vector of all ones.

In the rest of the paper we shall use $p = 1$, because the linear programming formulations are more practical if the decision variables are subject to integer restrictions.

We now review two special cases of interest for our problem.

4.2 Constraints with upper and lower bounds

The robust optimization paradigm implies that each constraint is immunized separately. However, the two sides of a two-sided inequality can be treated simultaneously. Indeed, let

$$\underline{b} \le \tilde{a}(\xi)^T x \le \bar{b} \tag{14}$$

be a two-sided inequality. It is easy to show that the same auxiliary (dual) variables t and w can be used to robustify the two sides of (14). Thus the set of vectors x satisfying the above two-sided inequality for all $\xi \in \Xi$ is equivalently written as follows:

$$\begin{aligned}
\bar{a}^T x + \kappa_1 t + e^T w &\le \bar{b} && \text{(15a)}\\
\bar{a}^T x - \kappa_1 t - e^T w &\ge \underline{b} && \text{(15b)}\\
te + w &\ge P^T x && \text{(15c)}\\
te + w &\ge -P^T x && \text{(15d)}\\
t \ge 0, w \ge 0. & && \text{(15e)}
\end{aligned}$$

The practical advantage of this formulation is that the above robust equivalent has only one more scalar inequality than (13).

4.3 Uncertainty factors with identical coefficients

In the capacity assignment problem, for some constraints, several random factors in $\{\xi_k\}_{k \in K}$ have the same coefficient. We propose to exploit this property.

As done previously we shall denote $z = P^T x$. Suppose that there exists a set of indices $J \subset \{1, \ldots, m\}$ such that the $z_i = z_l$ for all $i \in J$ and some $l \in J$. Denoting $I = \{1, \ldots, m\} \setminus J$, we have:

$$z^T \xi = \sum_{i \in I} z_i \xi_i + z_l \sum_{i \in J} \xi_i.$$

Furthermore, from Corollary 1, the robust equivalent is: $\bar{a}^T x + \sigma^*(x) \le b$, with:

$$\sigma^*(x) = \min_w \left\{ \kappa_1 \cdot \max \left\{ \max_{i \in I} |z_i - w_i|, \max_{i \in J} |z_i - w_i| \right\} + \sum_{i \in I} |w_i| + \sum_{i \in J} |w_i| \right\}.$$

It is easy to see that the condition $z_i = z_l$, for all $i \in J$, implies that the optimal solution w^* is such that the $w_i^* = w_l^*$ for all $i \in J$. Hence:

$$\sigma^*(x) = \min_w \left\{ \kappa_1 \cdot \max_{i \in I \cup \{l\}} |z_i - w_i| + \sum_{i \in I} |w_i| + \mathrm{card}(J) \cdot |w_l| \right\}.$$

Proposition 1 *Let $z = P^T x$. If there exists a set of indices $J \subset \{1, \ldots, m\}$ and $l \in J$ such that: $\forall i \in J, z_i = z_l$, then the robust equivalent for the robust constraint*

$$(\bar{a} + P\xi)^T x \le b, \ \forall \xi \in \Xi = \{\xi \mid \|\xi\|_\infty \le 1, \ \|\xi\|_1 \le \kappa_1\}$$

is the single constraint in variables x **and** *w*

$$\bar{a}^T x + \kappa_1 \cdot \max_{i \in I \cup \{l\}} |z_i - w_i| + \sum_{i \in I} |w_i| + \mathrm{card}(J) \cdot |w_l| \le b,$$

with $I = \{1, \ldots, m\} \setminus J$.

4.4 Probabilistic analysis

Corollary 1 does not rely on probabilistic arguments. Nevertheless, it can be given a probabilistic interpretation provided some assumptions are made on the behavior of the uncertainty factor ξ. The standard approach, as described in [6, chapter 2], consists in selecting a class of distributions with some known characteristics, e.g., a finite support and a known mean. The typical result consists in showing that for a well-defined value of κ_p, any solution of the equivalent robust counterpart (11) will satisfy the uncertain constraint with probability at least $1 - \epsilon$, whatever is the true distribution within the considered class.

In order to support the theoretical analysis with an empirical study, we need to simulate demand scenarios and thus use a probability distribution to generate them by Monte-Carlo sampling. For most experiments we chose the symmetric triangular distribution on $[-1, 1]$. This distribution is well-accepted in practice as reflecting a plausible behavior of the uncertain component. To be consistent with the simulation phase of the study, we based our theoretical estimate of constraint satisfaction on the same distribution. We could have chosen another element in the class of distributions with zero mean and range $[-1, 1]$. This would only impact the immunization factor κ_p. Roughly speaking, the less is known on the distribution, the larger is the immunization factor. On practical grounds, it means that its actual value is to be fixed by the decision maker on the basis of the simulation studies.

Assumption 2 *The random factors ξ_j are independent, with a symmetric triangular distribution on the interval $[-1, 1]$.*

Our probabilistic result is as follows.

Theorem 3 *Assume that the random variables ξ_i, $i = 1, \ldots, m$ are i.i.d. with a symmetric triangular distribution on the range $[-1, 1]$. If the equivalent robust counterpart (11) is satisfied with*

$$\kappa_p = \begin{cases} \sqrt{\frac{1}{3} \ln \frac{1}{\epsilon}} & \text{if} \quad p = 2 \\ \\ \sqrt{\frac{1}{3} \ln \frac{1}{\epsilon}} \sqrt{m} & \text{if} \quad p = 1, \end{cases}$$

then

$$\mathit{Prob}((\bar{a} + Px)^T \xi \leq b) \geq 1 - \epsilon.$$

Proof: For interested readers, the proof is given in the appendix. ■

Remark 2 *The factor $\sqrt{m}$ in the definition of κ_1 stems from the fact that the ℓ_1-ball with radius $\sqrt{m}$ is the ball with smallest radius that contains the unit ball in ℓ_2. Actually, the results pertaining to the ℓ_1-ball are consequences of the results pertaining to the ℓ_2-ball.*

Remark 3 *Similar results are obtained if Assumption 2 is modified to cover more general classes of distributions. We refer to [6, chap.2] for an extensive review of various classes. The main point is that the robust counterpart has almost the same structure as in Corollary 1. The difference is usually in the immunization factor κ_p. For instance, if one considers the class of distributions with support $[-1, 1]$ and mean $E(\xi) = 0$, Theorem 3 holds in its exact form with $\kappa_2 = \sqrt{2 \ln \frac{1}{\epsilon}}$ instead of $\kappa_2 = \sqrt{\frac{1}{3} \ln \frac{1}{\epsilon}}$, that is an increase by a factor almost 2.5. This increase is the price to pay to compensate for a partial information* *on the true distribution. In our experiments, we choose to work with Assumption 2 and perform simulations for the validation process under the same conditions. It can be proved that the immunization factor with the uniform distribution is $\kappa_2 = \sqrt{\frac{1}{2.5} \ln \frac{1}{\epsilon}}$. As expected, the factor is greater than in the case of the triangular distribution, because in the former case the distribution is more dispersed.*

5 Robust capacity assignment formulation

We now apply the concepts of robust optimization to the capacity assignment problem.

5.1 Demand constraints

The demand constraint requires that the total flow into a demand node meets the demand in a perfect equality. Since this demand constraint involves uncertain components, perfect equality cannot hold unless the equality is an identity with respect to each individual demand. In the case of the demand constraint (7c) associated with $k \in \mathcal{K}$, it means that the equation with uncertain coefficients

$$\sum_{i \in \mathcal{I}_k} (\alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik}\xi_k^v + \alpha_{3ik}\xi_k^r) = \bar{d}_k + \xi_k \hat{d}_k$$

holds for any possibly demand outcome only if the following equality constraints hold

$$\sum_{i \in \mathcal{I}_k} \alpha_{0ik} = \bar{d}_k, \ \sum_{i \in \mathcal{I}_k} \alpha_{1ik} = \hat{d}_k, \ \sum_{i \in \mathcal{I}_k} \alpha_{2ik} = 0, \ \sum_{i \in \mathcal{I}_k} \alpha_{3ik} = 0. \tag{16}$$

In formulation (16), each demand constraint is replaced by 4 equalities; we end up with $4|\mathcal{K}|$ equality constraints.

5.2 Capacity constraints

To obtain the robust counterpart of the capacity constraints (7b), we consider the polyhedral uncertainty set (10) with $p = 1$

$$\Xi^{cap} = \{\xi \mid \|\xi\|_\infty \leq 1, \ \|\xi\|_1 \leq \kappa_1^{cap}\}$$

Hence applying Corollary 1 and Theorem 2, the equivalent robust counterpart of the single robust capacity constraint

$$\sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}_k} \pi_{ik}^a (\alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik} \sum_{k' \in V_k} \xi_{k'} + \alpha_{3ik} \sum_{k' \in R_k} \xi_{k'}) \leq c_a, \quad \forall \xi \in \Xi^{cap}. \tag{17}$$

with $a \in \mathcal{A}$, is the system of linear inequalities

$$\begin{aligned}
&\sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}_k} \pi_{ik}^a \alpha_{0ik} + \kappa_1^{cap} v_a^{cap} + \sum_{k' \in \mathcal{K}} u_{ak'}^{cap} \leq c_a, \\
&u_{ak}^{cap} + v_a^{cap} \geq \pm \Big(\sum_{i \in \mathcal{I}_k} \pi_{ik}^a \alpha_{1ik} + \sum_{k' \in V_k, i \in \mathcal{I}'_k} \pi_{ik'}^a \alpha_{2ik} + \sum_{k' \in R_k, i \in \mathcal{I}'_k} \pi_{ik'}^a \alpha_{3ik} \Big), k \in \mathcal{K} \\
&u^{cap} \geq 0, \ v^{cap} \geq 0.
\end{aligned}$$

Finally the robust version of the capacity constraints has $2|\mathcal{A}| \times |\mathcal{K}|$ additional constraints and $|\mathcal{A}|(1 + |\mathcal{K}|)$ additional variables.

In Section 6, we experiment different values for κ_1^{cap} leading to different probabilities of satisfaction with Theore3.

5.3 Constraints of flow bounds

The ADR defines the flow in the recourse stage of the problem. This flow has to satisfy, for any demand $k \in \mathcal{K}$ and $i \in \mathcal{I}_k$:

$$y_{ik}M_k \geq \alpha_{0ik} + \alpha_{1ik}\xi_k + \alpha_{2ik}\xi_k^v + \alpha_{3ik}\xi_k^r \geq 0, \ \ \forall \xi \in \Xi^{pos} \tag{18}$$

with

$$\Xi^{pos} = \{\xi \mid \|\xi\|_\infty \leq 1, \ \|\xi\|_1 \leq \kappa_1^{pos}\}.$$

Note that the above constraint has identical coefficients for uncertainty factors and is a two sided inequality. Thus according to Proposition 1 and transformation rules (15) , the robust counterpart of (7d) is given by the following set of constraints:

$$\alpha_{0ik} - \kappa_1^{pos} v_{ik}^{pos} - u_{ik}^{pos} - n_k^v u_{kv}^{pos} - n_k^r u_{kr}^{pos} \geq 0 \tag{19a}$$
$$\alpha_{0ik} + \kappa_1^{pos} v_{ik}^{pos} + u_{ik}^{pos} + n_k^v u_{kv}^{pos} + n_k^r u_{kr}^{pos} \leq y_{ik}M_k \tag{19b}$$
$$u_{ik}^{pos} + v_{ik}^{pos} \geq \pm\alpha_{1ik} \tag{19c}$$
$$u_{ikv}^{pos} + v_{ik}^{pos} \geq \pm\alpha_{2ik} \tag{19d}$$
$$u_{ikr}^{pos} + v_{ik}^{pos} \geq \pm\alpha_{3ik} \tag{19e}$$
$$u^{pos} \geq 0, \ v^{pos} \geq 0. \tag{19f}$$

The above formulation of the flow bounds yields $6|\mathcal{I}|$ additional constraints and $4|\mathcal{I}|$ additional variables where $|\mathcal{I}|$ is the total number of paths.

In the experiments below, we set κ_1^{pos} such that it ensures from Theorem 3 a level of satisfaction for each constraint close to 100%. Note that the nonnegativity constraints are made robust with respect to larger uncertainty set (i.e., larger immunization factor) than the capacity constraints. Indeed it makes no sense to deal with negative flows on the network, whereas capacity violations can be interpreted as unsatisfied demand.

6 Numerical experiments

6.1 Test problems

The set of test problems consists of seven oriented graphs of various sizes. All of them correspond to physical networks, except `planar30` that was generated with a tool designed to produce network instances that simulate telecommunications problems. The corresponding data (topology, costs and demands) are publicly available either from `http://sndlib.zib.de/home.action` or from `http://www.di.unipi.it/di/groups/optimize/Data/MMCF.html` (in the case of `planar30`). The networks `polska`, `nobel-us`, `atlanta` and `france` belongs to the SNDlib; they are undirected. To make them directed, we substituted to each arc in the graphs two arcs with opposite directions. This choice guarantees that there always exists a feasible path for each OD pair.

For each problem instance, we have constructed a set of paths for each Origin-Destination (OD) pair according to the shortest distance (investment cost) criterion. To this end, we used the simple k-shortest paths algorithm [13]. Note that in some problem instances it may happen that for a few OD-pairs the number of distinct paths is less than k. Table 1 provides, in the first three columns, for each instance the number of nodes, the number of arcs and the number of origin-destination pairs of demands. The next column gives the total number of generated k-shortest paths with $k = 4$. The next two columns give an evaluation of the corresponding number of constraints and number of variables for the robust formulation (7) without integer constraints. The last two columns give the same information for the determistic formulation (2).

	#nodes	#arcs	#OD	#paths*	robust version (7) #const*	#var*	determin. version (2) #const*	#var*
`pdh`	11	34	24	54	2140	1316	146	88
`di-yuan`	11	42	22	61	2405	1496	167	103
`polska`	12	36	66	264	6900	4560	402	300
`nobel-us`	14	42	91	364	10598	6818	539	406
`planar30`	30	150	92	368	30694	17044	760	518
`atlanta`	15	44	210	840	25244	16048	1138	884
`france`	25	90	300	1188	63606	36684	1668	1278

(*) Indicative values using the k-shortest paths algorithm with $|\mathcal{I}_k| = 4$.
The nonnegativity constraints are included in the figures.

Table 1: Test problems.

Finally, the traffic demands are all uncertain with a range of variation ±50% of the nominal demand $\bar{d}_k$. This range of variation fits the observed yearly forecast errors.

6.2 Experiments

For each problem instance we considered four different models. They differ by the number of admissible paths per OD and the restriction on the number of usable paths per OD. The four models are as follows:

Model 1 The set of admissible paths for each OD pair reduces to a singleton ($|\mathcal{I}_k| = 1$). The flow is de facto restricted to a single path. (Results in Table 3.) In that case the ADR reduces to $f_k = d_k$. the problem is simple, but not deterministic.

Model 2 The set of admissible paths for each OD pair consists of four paths ($|\mathcal{I}_k| = 4$) and there is no restriction on the number of paths on which the flow can be routed. (Results in Table 4.)

Model 3 The set of admissible paths for each OD pair consists of four paths ($|\mathcal{I}_k| = 4$), but the flow is restricted to be routed on a single path. (Results in Table 5.)

Model 4 The set of admissible paths for each OD pair consists of four paths ($|\mathcal{I}_k| = 4$), but the flow is restricted to be routed on 2 paths. (Results in Table 6.)

For each model we performed experiments with different levels of risk $1-\epsilon$ for the capacity constraints, but the a fixed level of risk for the non-negativity constraints. For the latter we use $1-\epsilon = 99.75\%$; for the former we use the four values $1-\epsilon \in \{85\%, 50\%, 10\%, 5\%\}$. According to Theorem 3 each level of risk defines an immunization coefficient $\kappa_1 = \sqrt{\frac{1}{3}\ln\frac{1}{\epsilon}} \times \sqrt{m}$, where m is the dimension of the random factor, that is m is equal to the number of OD pairs (if all demands are uncertain). The κ_1^{cap} and κ_1^{pos} values are reported in Table 2.

Note that Model 1 and 3 both use a single path per OD, but in Model 1 the path is imposed, while in Model 3 its selection is to be made among four possible candidates by the optimization process. All instances were solved with Cplex 12.2 (using the *baropt* option) on a computer running an Intel(R) Xeon(TM) processor 2.80GHz with 3Gb of RAM. A time limit of 5 hours was imposed on each instance to solve robust capacity expansion problem.

	$1-\epsilon$ for capacity constraints				
	99.75%	85%	50%	10%	5%
pdh	6.93	3.89	2.35	0.92	0.64
di-yuan	6.63	3.73	2.25	0.88	0.61
polska	11.49	6.46	3.91	1.52	1.06
nobel-us	13.49	7.59	4.59	1.79	1.25
planar30	13.56	7.63	4.61	1.80	1.25
atlanta	20.49	11.52	6.97	2.72	1.89
france	24.49	13.77	8.33	3.25	2.26

Table 2: κ_1^{cap} and κ_1^{pos}values assuming triangular distributions.

Remark 4 *The limited number of admissible paths for each OD pair (i.e., $|\mathcal{I}_k| = 4$) is not restrictive in practice. Decision planners are not willing to take into consideration many alternatives for routing the traffic.*

6.3 Validation process

The goal of the validation process is to provide an empirical evaluation of the risk associated with a solution for the capacity assignment problem. This is done by generating a sample of demand scenarios and analyzing the performance of the selected solution on each scenario. For each problem instance, we build a sample of 1000 simulations. The simulations are performed by a Monte-Carlo scheme in accordance with Assumption 2, that is, the demands d_k are assumed to be independent with a symmetrical triangular distribution on $[\bar{d}_k - \hat{d}_k, \bar{d}_k + \hat{d}_k]$, with $\hat{d}_k = 0.5\bar{d}_k$. At each simulation, a random matrix is sampled from the distributions. Recall that an optimal solution is a pair of installed capacities c^* and affine decision rules for the flows $f^*(d)$. The rules are such that the demand constraints $\sum_{i\in\mathcal{I}_k^*} f_{ik}^*(d) = d_k$ are satisfied by construction. This is not true for the capacity constraints $\sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}_k^*} f_{ik}^*(d)\pi_{ik}^a \leq c_a^*$. Consequently, some constraints may be violated for some simulated demand scenarios. However, one must check whether the constraint violation is intrinsic to the scenarios or is only due to a failure of the robust ADRs. Recall that the ADRs are only sub-optimal. It may very well happen that there exists an alternative routing that is feasible when the one prescribed by the ADR solution is not. The computations in the simulation process are organized as follows. We first check whether the ADR solution is feasible—a numerically easy check. If not, we solve the auxiliary multicommodity flow problem

$$\min_{f\geq 0, s\geq 0} \quad \sum_{k\in\mathcal{K}} s_k \tag{20a}$$

$$\sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}_k^*} f_{ik}\pi_{ik}^a \leq c_a^*, \quad a \in \mathcal{A} \tag{20b}$$

$$\sum_{i\in\mathcal{I}_k^*} f_{ik} = d_k - s_k, \quad k \in \mathcal{K}, \tag{20c}$$

where c^* is the optimal capacity assignment vector and I_k^* is the set of paths selected in the optimization process. If the objective value is positive, the capacities and the selected set of paths do not allow demand satisfaction in full. We denote $E_s = \sum_{k\in\mathcal{K}} s_k$ the amount of unserved demand (i.e., the excess amount of traffic).

6.4 Results

The results for Model 1 to 4 are reported in Tables 3 to 6. In those tables, for each instance we report the values of the robust solutions with different immunization factors and for the solutions of two deterministic problems: one with demands fixed at their mean value $\bar{d}$ (problem which is referred to as "deterministic solution" in the result tables) and one with demands fixed at their maximal value. The latter amounts to "total protection". Note that the total protection solution is also the one that is delivered by the robust model when the immunization factor is very large. Tables 5 and 6 pertaining to Model 3 and 4 do not involve the two largest instances `atlanta` and `france`, because the solution time for solving the robust formulation of the capacity expansion problem exceeded the time limit (5 hours). (Note that this time does not include the simulation process.) The difficulty stems from the complexity induced by the integral restriction on the number of paths (1 or 2 among 4) to be used for each OD.

The results for a particular problem are displayed on four rows in each table. The first row displays the solution cost ("Solutions"); the second row displays the proportion of simulations for which the traffic to be routed exceeds the installed capacity ("% of violations"); the third row shows the average conditional excess traffic in terms of fraction of lost traffic ("Rel. cond. viol."). The average conditional excess traffic is computed as follows. For each simulation s, let D_s be the total traffic to be routed, and E_s the excess amount of traffic. Let $\bar{S}$ denote the number of simulations for which a violation exists (i.e. $E_s > 0$). Then, the average conditional excess traffic is given by: $1/\bar{S} \cdot \sum_s E_s/D_s$. Finally, the last row gives the unconditional average amount of lost traffic ("Expected traffic loss"). We could have dispensed with this last row, as it only displays the product of the two preceding rows. We chose to include them to facilitate the evaluation.

We comment here on some specific points worth noticing.

Computational issues We present results for the Models 1 to 4. The first two models involve continuous variables only, while Models 3 and 4 involve binary variables also. When robust optimization is applied to Models 1 and 2 in continuous variables, the problem dimension increases in a controlled way and computational times remain reasonable. For instance, robust formulations of the largest instances `planar30`, `atlanta` and `france` are solved in about 1, 1.5 and 5 minutes respectively. Even though the deterministic versions of these problems are solved in a few seconds, the increase in computational time for their robust counterpart is acceptable.

When the number of paths for each commodity is further restricted to be integer, as in Models 3 and 4, the problems are much harder to solve. Within a time limit of 5 hours, we could solve most instances, but the two largest ones (`atlanta` and `france`). Interestingly enough, the solutions on the solved instances display performances that lie in-between those for Models 1 and 2. Because the focus of the paper is on the handling of uncertainty, we shall concentrate our next comments on Models 1 and 2.

Controlling feasibility The percentage of violations is always much smaller than the targeted risk probability ϵ obtained from the theoretical analysis of Theorem 3. We can invoke several reasons for this. First, the robust counterpart is built on the assumption that ADR are used, while in the validation process we use fully adjustable recourse (when ADR lead to infeasible solutions). The latter may be feasible, while the former turn out to be not. (See Table 9 and 10 for a refined analysis.) Second, we must recall that Theorem 3 only provides a lower bound on the probability of constraint satisfaction. The theorem makes it possible to interpret the equivalent robust counterpart as a "safe tractable approximation" of the chance constraint (see

	Total Protection	Robust solutions with $1-\epsilon$ for capacity constraints 85%	50%	10%	5%	Deterministic solution
			pdh			
Solutions	7.52E+008	7.52E+008	7.52E+008	7.32E+008	6.62E+008	5.02E+008
% of violations	0.00%	0.00%	0.00%	7.50%	79.30%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.06%	0.47%	8.24%
Expected traffic loss	0.00%	0.00%	0.00%	0.00%	0.37%	8.24%
			di-yuan			
Solutions	5.95E+006	5.94E+006	5.74E+006	5.20E+006	4.82E+006	3.97E+006
% of violations	0.00%	0.00%	0.10%	32.60%	81.80%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.15%	0.53%	0.95%	7.42%
Expected traffic loss	0.00%	0.00%	0.00%	0.17%	0.78%	7.42%
			polska			
Solutions	7.04E+006	6.90E+006	6.55E+006	5.66E+006	5.41E+006	4.69E+006
% of violations	0.00%	0.00%	0.00%	41.20%	81.70%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.39%	0.63%	4.32%
Expected traffic loss	0.00%	0.00%	0.00%	0.16%	0.51%	4.32%
			nobel-us			
Solutions	1.29E+008	1.28E+008	1.23E+008	1.08E+008	1.03E+008	8.60E+007
% of violations	0.00%	0.00%	0.00%	15.00%	58.50%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.19%	0.29%	4.51%
Expected traffic loss	0.00%	0.00%	0.00%	0.03%	0.17%	4.51%
			planar30			
Solutions	6.61E+007	6.44E+007	6.13E+007	5.34E+007	5.07E+007	4.31E+007
% of violations	0.00%	0.00%	0.00%	39.50%	84.00%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.42%	0.66%	5.20%
Expected traffic loss	0.00%	0.00%	0.00%	0.17%	0.55%	5.20%
			atlanta			
Solutions	4.59E+011	4.53E+011	4.41E+011	4.04E+011	3.87E+011	3.07E+011
% of violations	0.00%	0.00%	0.00%	0.40%	5.60%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.11%	0.12%	3.90%
Expected traffic loss	0.00%	0.00%	0.00%	0.00%	0.01%	3.90%
			france			
Solutions	6.65E+008	6.42E+008	6.10E+008	5.43E+008	5.22E+008	4.44E+008
% of violations	0.00%	0.00%	0.00%	1.90%	16.10%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.10%	0.12%	3.07%
Expected traffic loss	0.00%	0.00%	0.00%	0.00%	0.02%	3.07%

Table 3: Robust solutions and performances for Model 1 ($|\mathcal{I}_k| = 1$: only one shortest path).

	Total Protection	Robust solutions with $1-\epsilon$ for capacity constraints 85%	50%	10%	5%	Deterministic solution
			pdh			
Solutions	7.52E+008	7.52E+008	7.52E+008	7.00E+008	6.49E+008	5.02E+008
% of violations	0.00%	0.00%	0.00%	30.10%	72.40%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.53%	0.67%	8.42%
Expected traffic loss	0.00%	0.00%	0.00%	0.16%	0.49%	8.42%
			di-yuan			
Solutions	5.95E+006	5.81E+006	5.59E+006	5.07E+006	4.74E+006	3.97E+006
% of violations	0.00%	0.00%	0.50%	24.20%	75.10%	100.00%
Rel. cond. viol.	0.00%	0.00%	1.54%	0.64%	1.04%	7.30%
Expected traffic loss	0.00%	0.00%	0.01%	0.15%	0.78%	7.30%
			polska			
Solutions	7.00E+006	6.56E+006	6.10E+006	5.44E+006	5.27E+006	4.69E+006
% of violations	0.00%	0.00%	0.10%	16.40%	34.90%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.48%	0.51%	0.58%	3.97%
Expected traffic loss	0.00%	0.00%	0.00%	0.08%	0.20%	3.97%
			nobel-us			
Solutions	1.28E+008	1.22E+008	1.16E+008	1.03E+008	9.92E+007	8.60E+007
% of violations	0.00%	0.00%	0.00%	1.60%	12.30%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.25%	0.27%	4.39%
Expected traffic loss	0.00%	0.00%	0.00%	0.00%	0.03%	4.39%
			planar30			
Solutions	6.54E+007	6.25E+007	5.91E+007	5.22E+007	4.99E+007	4.31E+007
% of violations	0.00%	0.00%	0.00%	17.10%	57.70%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.31%	0.45%	5.13%
Expected traffic loss	0.00%	0.00%	0.00%	0.05%	0.26%	5.13%
			atlanta			
Solutions	4.53E+011	4.41E+011	4.27E+011	3.93E+011	3.77E+011	3.07E+011
% of violations	0.00%	0.00%	0.00%	0.10%	3.20%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.34%	0.11%	3.88%
Expected traffic loss	0.00%	0.00%	0.00%	0.00%	0.00%	3.88%
			france			
Solutions	6.18E+008	5.86E+008	5.56E+008	5.08E+008	4.92E+008	4.44E+008
% of violations	0.00%	0.00%	0.00%	0.90%	9.80%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.16%	0.14%	3.10%
Expected traffic loss	0.00%	0.00%	0.00%	0.00%	0.01%	3.10%

Table 4: Robust solutions and performances for Model 2 ($|\mathcal{I}_k| = 4$ and all 4 shortest paths can support the solution).

	Total Protection	Robust solutions with $1-\epsilon$ for capacity constraints 85%	50%	10%	5%	Deterministic solution
			pdh			
Solutions	7.52E+008	7.52E+008	7.52E+008	7.29E+008	6.62E+008	5.02E+008
% of violations	0.00%	0.00%	0.00%	24.00%	79.90%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.28%	0.49%	8.27%
Expected traffic loss	0.00%	0.00%	0.00%	0.07%	0.39%	8.27%
			di-yuan			
Solutions	5.96E+006	5.88E+006	5.67E+006	5.13E+006	4.78E+006	3.97E+006
% of violations	0.00%	0.00%	0.10%	31.30%	81.00%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.43%	0.63%	0.93%	7.43%
Expected traffic loss	0.00%	0.00%	0.00%	0.20%	0.75%	7.43%
			polska			
Solutions	7.04E+006	6.85E+006	6.42E+006	5.61E+006	5.37E+006	4.69E+006
% of violations	0.00%	0.00%	0.10%	27.10%	62.60%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.04%	0.36%	0.50%	4.32%
Expected traffic loss	0.00%	0.00%	0.00%	0.10%	0.31%	4.32%
			nobel-us			
Solutions	1.29E+008	1.27E+008	1.21E+008	1.06E+008	1.01E+008	8.60E+007
% of violations	0.00%	0.00%	0.00%	5.80%	34.30%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.21%	0.24%	4.53%
Expected traffic loss	0.00%	0.00%	0.00%	0.01%	0.08%	4.53%
			planar30			
Solutions	6.60E+007	6.38E+007	6.05E+007	5.30E+007	5.04E+007	4.31E+007
% of violations	0.00%	0.00%	0.10%	44.80%	81.00%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.39%	0.43%	0.64%	5.18%
Expected traffic loss	0.00%	0.00%	0.00%	0.19%	0.52%	5.18%

Table 5: Robust solutions and performances for Model 3 (only one path among the four shortest paths can be used by the solution).

	Total Protection	Robust solutions with $1-\epsilon$ for capacity constraints 85%	50%	10%	5%	Deterministic solution
			pdh			
Solutions	7.52E+008	7.52E+008	7.52E+008	7.02E+008	6.50E+008	5.02E+008
% of violations	0.00%	0.00%	0.00%	30.40%	78.60%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.41%	0.72%	8.26%
Expected traffic loss	0.00%	0.00%	0.00%	0.12%	0.57%	8.26%
			di-yuan			
Solutions	5.95E+006	5.81E+006	5.59E+006	5.07E+006	4.74E+006	3.97E+006
% of violations	0.00%	0.00%	0.10%	30.70%	74.90%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.84%	0.69%	1.06%	7.32%
Expected traffic loss	0.00%	0.00%	0.00%	0.21%	0.79%	7.32%
			polska			
Solutions	7.01E+006	6.58E+006	6.13E+006	5.45E+000	5.27E+006	4.69E+006
% of violations	0.00%	0.00%	0.00%	14.80%	30.20%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.50%	0.60%	4.39%
Expected traffic loss	0.00%	0.00%	0.00%	0.07%	0.18%	4.39%
			nobel-us			
Solutions	1.28E+008	>5h	>5h	1.03E+008	9.94E+007	8.60E+007
% of violations	0.00%	-	-	2.40%	7.90%	100.00%
Rel. cond. viol.	0.00%	-	-	0.22%	0.29%	4.55%
Expected traffic loss	0.00%	-	-	0.01%	0.02%	4.55%
			planar30			
Solutions	6.54E+007	6.26E+007	5.92E+007	5.23E+007	5.00E+007	4.31E+007
% of violations	0.00%	0.00%	0.00%	20.10%	55.70%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.00%	0.35%	0.43%	5.12%
Expected traffic loss	0.00%	0.00%	0.00%	0.07%	0.24%	5.12%

Table 6: Robust solutions and performances for Model 4 (at most two paths among the four shortest paths can be used by the solution).

[6]) at the expense of some conservativeness. A third reason for the discrepancy stems from the fact that the analysis treats the constraints as independent elements. This is certainly not the case. To handle the issue of global robustness in a model, one would have to replace the separate capacity constraints as a single one (for instance by using the maximum of auxiliary surplus variables). This approach would provide a stronger result, but it leads to an intractable robust counterpart. Clearly, to work with numerically tractable models, we have to live with the fact that the robust solution is by construction conservative. This is why it is suitable to use simulations to assess the experimental risk associated with the solutions.

The gap between the theoretical guarantee and the simulated feasibility tends to be larger for big networks, see instances `atlanta` and `france` in Table 3 and 4. Observe that for all instances, targeting theoretically a 50% feasibility ($1-\epsilon = 50\%$) still leads to solutions with almost no violated simulation.

Cost analysis Table 7 displays the saving in investment cost as a function of the risk level for Model 1. The saving is measured in terms of a (negative) percentage of the total protection cost. Table 7 (Model 1 with only one available path per OD pair) exhibits an average gain of 5% for a theoretical level $1-\epsilon = 50\%$ with almost zero violation across the simulations. We conclude that the 5% saving is achieved at no risk. If we use the conditional and unconditional average traffic loss as an alternative measure of risk, we observe through Tables 3-6 that those values never exceed 0.69% and 0.21% (and are much lower in most cases) for the solution with a theoretical level $1-\epsilon = 10\%$. If these traffic loss figures are deemed acceptable, then the solution with $1-\epsilon = 10\%$ allows for a gain of 14% in the average (Table 7).

Table 8 displays the gain in the overall investment cost when using up to four routings per demand (Model 2) instead of one (Model 1). Hence, this table gives the impact of path diversification. The gain can be large (up to 8.8% for network `france`, when $1-\epsilon = 50\%$). For the cases of practical interest ($1-\epsilon = 50\%$ or $1-\epsilon = 10\%$), the average gain related to path diversification is roughly 4%.

To better illustrate the above comments, Figure 1 displays the evolution of the cost with the expected traffic loss for four representative instances (`diyuan`, `polska`, `nobel-us` and `france`).

Impact of affine decision rules Affine decision rules (ADR) are used to control the numerical complexity of our models, but they are restrictive. It turns out that in some simulations the ADR violates the capacity constraints, while a feasible routing can be shown to exist. To capture the negative impact of the ADR, we use Model 2 (with all four paths available) and report two indicators. Table 9 gives the proportion of simulations for which the ADR meets the capacity constraints. With a theoretical risk level $1-\epsilon \geq 50\%$, the ADR solution meets the requirements in nearly 98% cases. Table 10 complements these results with an information on the average number of simulations for which the ADR fails (i.e., value smaller than 100% in Table 9), but a feasible routing has been found. This number increases with $1-\epsilon$, but usually remains low when $1-\epsilon \geq 50\%$. This is a good indication that ADR perform very well, at least when the taken risk is moderate.

6.5 Additional numerical experiments

The additional numerical experiments in this section focus on the particular instance `polska` and use Model 2 ($|\mathcal{I}_k| = 4$ and all 4 shortest paths can support the solution).

	Total	1-ϵ				Deterministic
	protection	85%	50%	10%	5%	solution
pdh	0%	0%	0%	-3%	-12%	-33%
diyuan	0%	0%	-3%	-13%	-19%	-33%
polska	0%	-2%	-7%	-20%	-23%	-33%
nobel-us	0%	-1%	-4%	-16%	-20%	-33%
planar30	0%	-3%	-7%	-19%	-23%	-33%
atlanta	0%	-1%	-4%	-12%	-16%	-33%
france	0%	-3%	-8%	-18%	-22%	-33%
average	0%	-2%	-5%	-14%	-19%	-33%

Table 7: Decrease of the investment cost with respect to the cost for total protection, with Model 1 (only one shortest path).

	Total	1-ϵ				Deterministic
	protection	85%	50%	10%	5%	solution
pdh	0.00%	0.00%	0.00%	-4.37%	-1.92%	0.00%
diyuan	0.00%	-2.14%	-2.66%	-2.42%	-1.60%	0.00%
polska	-0.54%	-4.93%	-6.80%	-3.78%	-2.64%	0.00%
nobel-us	-0.62%	-4.23%	-6.09%	-4.45%	-3.40%	0.00%
planar30	-1.01%	-2.87%	-3.46%	-2.28%	-1.52%	0.00%
atlanta	-1.42%	-2.61%	-3.18%	-2.80%	-2.41%	0.00%
france	-7.08%	-8.81%	-8.80%	-6.57%	-5.60%	0.00%
average	-1.52%	-3.66%	-4.43%	-3.81%	-2.73%	0.00%

Table 8: Decrease of the investment cost with Model 2 (routings allowed on the 4 shortest paths), with respect to Model 1 (routing on the unique shortest path).

	Total	1-ϵ				Deterministic
	protection	85%	50%	10%	5%	solution
pdh	100.00%	100.00%	100.00%	52.50%	17.10%	100.00%
diyuan	100.00%	100.00%	98.20%	40.70%	15.90%	100.00%
polska	100.00%	99.70%	90.80%	22.80%	7.60%	100.00%
nobel-us	100.00%	99.90%	97.10%	37.10%	10.40%	100.00%
planar30	100.00%	100.00%	97.60%	28.70%	5.70%	100.00%
atlanta	100.00%	100.00%	100.00%	75.20%	42.70%	100.00%
france	100.00%	100.00%	99.30%	50.30%	10.00%	100.00%
average	100.00%	99.94%	97.57%	43.90%	15.63%	100.00%

Table 9: Percentage of simulated data for which ADR are sufficient to route flows. These results are obtained for Model 2 (all 4 shortest paths can support the solution).

	Total	1-ϵ				Deterministic
	protection	85%	50%	10%	5%	solution
pdh	0.00%	0.00%	0.00%	17.40%	10.50%	0.00%
diyuan	0.00%	0.00%	1.30%	35.10%	9.00%	0.00%
polska	0.00%	0.30%	9.10%	60.80%	57.50%	0.00%
nobel-us	0.00%	0.10%	2.90%	61.30%	77.30%	0.00%
planar30	0.00%	0.00%	2.40%	54.20%	36.60%	0.00%
atlanta	0.00%	0.00%	0.00%	24.70%	54.10%	0.00%
france	0.00%	0.00%	0.70%	48.80%	80.20%	0.00%
average	0.00%	0.06%	2.34%	43.19%	46.46%	0.00%

Table 10: Percentage of simulated data for which ADR do not enable to route flows, while there is a solution. These results are obtained for Model 2 (routings are allowed on the $|\mathcal{I}_k| = 4$ shortest paths).

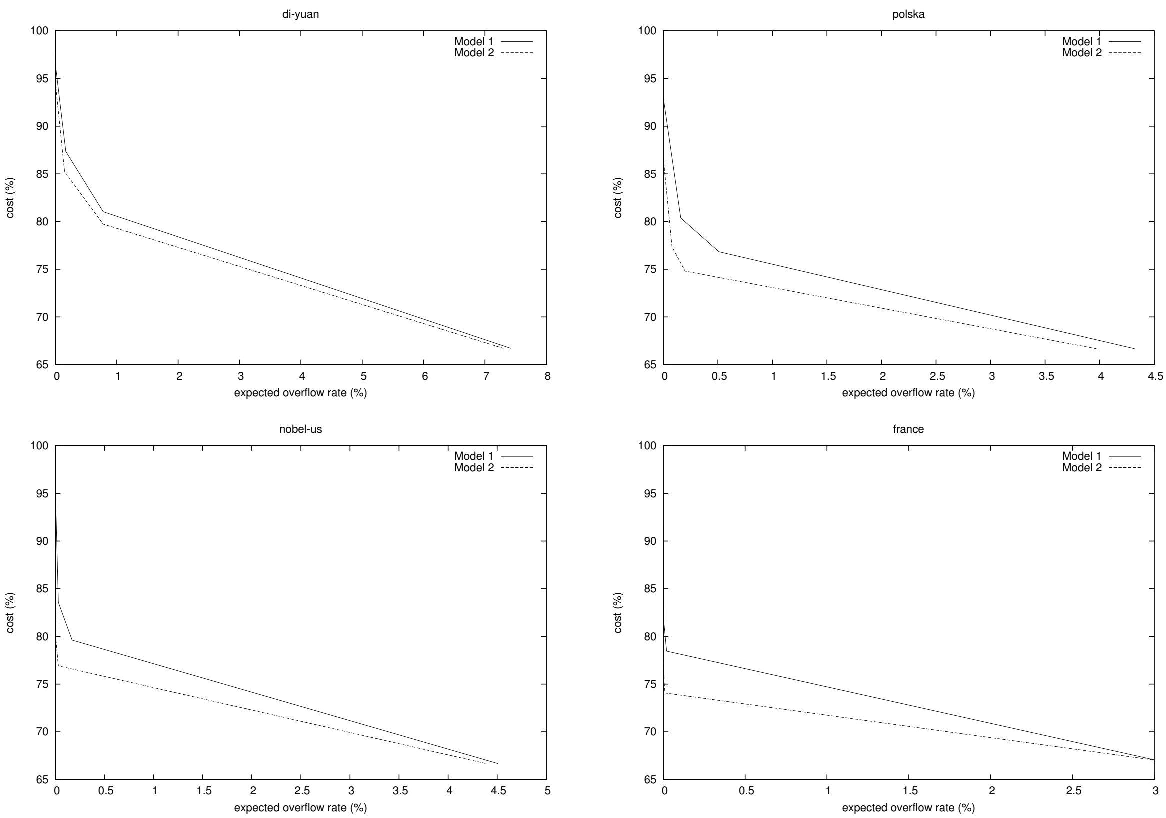

Figure 1: Evolution of the investment cost with the expected traffic loss for instances `diyuan`, `polska`, `nobel-us` and `france`. The 100%-cost corresponds to the case of total protection when routing for Model 1 (worst case).

Notwithstanding the choice of this particular instance, the results and the conclusions reported in this section are representative for all instances and models. In addition to the standard statistics used previously, we display in the following tables the maximum observed violation.

6.5.1 Comparison with a naive approach

We compare here the performances of the robust solutions with the solution performances obtained with a naive approach. The naive approach simply consists of considering a uniform proportion of the maximum possible demand for all OD-pairs and of solving the deterministic formulation with those demands. In other words, the naive approach replaces the uncertain demand quantities with a certain percentage of the maximum demand, and then solves a deterministic problem. To make the comparison consistent we use the proportions leading to the same cost solutions computed with the robust optimization technique. For example, in Table 11, in the second column we observe that to attain an objective cost of $6.56E+006$ a $1-\epsilon = 85\%$ is needed for the robust approach whereas the naive approach solves the deterministic problem with a level of 93.2% of the maximum possible demands. Those levels of maximum possible demands are computed empirically. Note that the 100% value in the naive approach corresponds to the total protection case in the previous sections.

Cost solutions	6.56E+006	6.10E+006	5.44E+006	5.27E+006
	Performances of robust solutions Robust solutions with $1-\epsilon$ for capacity constraints			
	85%	50%	10%	5%
% of violations	0.00%	0.10%	16.40%	34.90%
Rel. cond. viol.	0.00%	0.48%	0.51%	0.58%
Expected traffic loss	0.00%	0.00%	0.08%	0.20%
Violation max.	0.00%	0.63%	2.19%	2.79%
	Performances of naive solutions			
% of max demand	93.2%	86.6%	77.3%	74.9%
% of violations	0.05%	18.00%	85.00%	91.00%
Rel. cond. viol.	0.08%	0.34%	0.51%	0.94%
Expected traffic loss	0.00%	0.06%	0.43%	0.86%
Violation max.	0.16%	0.53%	2.9%	2.9%

Table 11: Performances with a naive approach with instance `polska` and Model 2 ($|\mathcal{I}_k| = 4$ and all 4 shortest paths can support the solution).

Table 11 shows that, for a similar cost level, the robust approach allows to reduce significantly the proportion of simulations with demand violation and thus the total expected traffic loss but it has no real impact on the magnitude of the relative conditional violation and of the maximum violation.

6.5.2 Testing alternative definitions of demand proximity

In this section, we experiment the full ADR formulation (4), i.e., without using demand proximity for ADR, and different definitions of demand proximity:

Proximity definition 1 The traffic demand k' is close to the demand k if their respective shortest paths have at least one arc in common. (This definition is the one that has been used in the preceding experiments.)

Proximity definition 2 The traffic demand k' is close to the demand k if their respective available paths have at least one arc in common.

Proximity definition 3 The traffic demand k' is close to the demand k if they have the same origin and/or destination.

Proximity definition 4 All traffic demand k' are close to the demand k.

Table 12 displays the investment cost performances for the representative instance `polska` and Model 2, when different proximity definitions are used. We observe that for a same level of constraint satisfaction, the full ADR improves on the best proximity definition (Definition 1). The price to pay for the better result of the full ADR is a significant increase of the number of continuous variables and of the computing time. On the instance `polska` the number of continuous decision variables (i.e., the number of parameters in the ADR) is multiplied by 22 (from 3 to 66) and the computing time is about 5 times larger. Note that the increase of time is not excessive on this instance as `polska` is a small problem solved in few seconds using proximity definitions. On the larger instance `france` the number of variables would be multiplied by a factor 100, making the MIP computation problematic. Using rules based on proximity is thus a reasonable compromise.

	1-ϵ			
	85%	50%	10%	5%
Full ADR	6.50E+006	5.99E+006	5.32E+006	5.16E+006
Proximity definition 1	6.56E+006	6.10E+006	5.44E+006	5.27E+006
Proximity definition 2	6.74E+006	6.25E+006	5.50E+006	5.30E+006
Proximity definition 3	6.73E+006	6.23E+006	5.49E+006	5.29E+006
Proximity definition 4	6.74E+006	6.25E+006	5.50E+006	5.30E+006

Table 12: Cost performances for different proximity definitions on instance `polska` and Model 2 ($|\mathcal{I}_k| = 4$ and all 4 shortest paths can support the solution).

6.5.3 Testing a uniform distribution assumption in the validation process

As pointed out in Remark 3, the robust counterparts of the uncertain constraints are the same, up to the immunization factor, for all distributions of the uncertain factor that are symmetric on a given range. It is thus interesting to perform the validation process not only for the symmetric triangular distribution but also the uniform distribution. Table 13 provides the results of the comparative study of both distribution assumptions, performed on Model 2 with the instance `polska`. In this table, the immunization factors are the same as those selected in the previous studies. For each value of κ_1^{cap} considered in the table, the corresponding theoretical protection level $(1-\epsilon)$ is given for both distribution assumptions. It is of course lower for the uniform distributions than for the triangular ones.

The empirical violations are higher in the validation process using a uniform distribution. This is not surprising, because the variability of the uniform distribution is greater than the symmetric triangular distribution. These results exhibit a pattern that is comparable with the one with the triangular distribution and thus the same conclusions apply. In view of Remark 3, one can assess the level of protection the theory predicts when

	Total Protection	Robust solutions with κ_1^{cap} for capacity constraints 6.46	3.91	1.52	1.06	Deterministic solution
Robust solutions for instance `polska`						
Solutions	7.00E+006	6.56E+006	6.10E+006	5.44E+006	5.27E+006	4.69E+006
Triangular distribution						
Protection $1-\epsilon$ (theory)	100%	85%	50%	10%	5%	
Empirical performances						
% of violations	0.00%	0.00%	0.10%	16.40%	34.90%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.48%	0.51%	0.58%	3.97%
Expected traffic loss	0.00%	0.00%	0.00%	0.08%	0.20%	3.97%
Violation max.	0.00%	0.00%	0.63%	2.19%	2.79%	8.51%
Uniform distribution						
Protection $1-\epsilon$ (theory)	100%	79%	44%	8%	4%	
Empirical performances						
% of violations	0.00%	0.00%	2%	46.00%	62.00%	100.00%
Rel. cond. viol.	0.00%	0.00%	0.56%	1.08%	1.15%	6.26%
Expected traffic loss	0.00%	0.00%	0.01%	0.50%	0.71%	6.26%
Violation max.	0.00%	0.00%	0.84%	3.72%	4.7%	11.50%

Table 13: Performances using triangular and uniform distributions for instance `polska` and Model 2 ($|\mathcal{I}_k| = 4$ and all 4 shortest paths can support the solution).

one uses the immunization factors displayed on Table 13. The computed values for each distribution are reported in the table.

7 Conclusion

In the present study we implemented a robust optimization approach to cope with demand uncertainty in the capacity planning of telecommunications networks. The proposed methodology allows the decision maker to balance between the need of Quality of Service (QoS) and the investment costs. The main feature of the method is the use of a refined version of Affine Decision Rules (ADR) to model the recourse decisions, i.e., the flow routing, as linear functions of the revealed uncertainty. To reduce the dimension of the robust model induced by the ADRs, we used the concept of demand proximity. To calibrate the immunization factor κ in the equivalent robust counterpart of the uncertain constraints, we performed a probabilistic analysis under the assumption that each demand is distributed according to a triangular symmetric distribution. Finally we have tested our approach with numerical experiments on network instances from SNDlib and performed a validation analysis by simulation to study the quality of the solutions. The results obtained on the test problems suggest several interesting conclusions. First and contrarily to most of the classical approaches, the proposed robust formulations remain numerically tractable and even allow the introduction of integer constraints to bound the number of used paths. Second, although the introduction of refined ADR makes the recourse decisions very restrictive and conservative in theory, our numerical tests illustrate their relevance in practice. It appears that in most of the cases of practical significance, ADR are in fact not restrictive. The performances of the computed solutions are convincing; the investment cost savings are significant and the QoS very high.

For future research we believe that the proposed methodology can be improved using the concept of globalized robustness [6] to control the magnitude of possible violations. Actually, our approach concentrates on solutions that remain feasible for all realizations within the uncertainty sets, but is silent about realizations that lie outside. Globalized robust optimization proposes an extension that admits possible constraint violations, but control their magnitude.

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A Proof of Theorem 3

We prove the theorem with $p = 2$ first. For the sake of simpler notations, we first rewrite the uncertain constraint (9) as

$$z_0 + z^T \xi \leq 0, \tag{21}$$

with $z_0 = \bar{a}^T - b$ and $z = P^T x$. We denote $Z = z_0 + z^T \xi$.

From the Markov inequality we bound the probability of violation of (21) by $\text{Prob}(Z \geq 0) = \text{Prob}(e^Z \geq 1) \leq E(e^Z)$. Noting that $\text{Prob}(Z \geq 0) = \text{Prob}(tZ \geq 0)$ holds for all $t > 0$, we have the stronger bound

$$\text{Prob}(Z \geq 0) \leq \inf_{t>0} E(e^{tZ}).$$

Let us explain the right-hand side. Remembering that the ξ_j are independent random variables, we may write

$$E(e^{tZ}) = E(e^{t(z_0 + \sum_j z_j \xi_j)}) = e^{tz_0} E\left(\prod_j e^{tz_j \xi_j}\right) = e^{tz_0} \prod_j E(e^{tz_j \xi_j}).$$

We claim that the above quantity is bounded by $e^{\frac{t^2}{12}\sum_j z_j^2}$. To prove the claim, we focus on the typical component $E(e^{tz_j\xi_j})$. For the sake of simpler notation, we temporarily drop the index j and we denote $\tau = tz$. From the triangular distribution assumption on ξ, we have

$$\begin{aligned} E(e^{\tau\xi}) &= \int_{-1}^{0} e^{\tau\xi}(1+\xi)d\xi + \int_{0}^{1} e^{\tau\xi}(1-\xi)d\xi && \text{(22a)} \\ &= \frac{e^{\tau} + e^{-\tau} - 2}{\tau^2} && \text{(22b)} \\ &= \frac{1}{\tau^2}\left[\sum_{k=0}^{\infty}\frac{\tau^k}{k!} + \sum_{k=0}^{\infty}\frac{(-\tau)^k}{k!} - 2\right] && \text{(22c)} \\ &= 2\sum_{k=0}^{\infty}\frac{(\tau^2)^k}{(2k+2)!} = \sum_{k=0}^{\infty} a_k (\tau^2)^k, && \text{(22d)} \end{aligned}$$

with $a_k = 2/(2k+2)!$.

The next step consists in bounding $E(e^{\tau\xi})$ by

$$e^{\alpha^2\tau^2} = \sum_{k=0}^{\infty}\frac{(\alpha^2\tau^2)^k}{k!} = \sum_{k=0}^{\infty} b_k(\tau^2)^k, \tag{23}$$

with $b_k = \alpha^{2k}/k!$. To prove the claim, we compare the two series term-wise and show that $a_k \leq b_k$ for all k Clearly, $a_0/b_0 = 1$. For $k = 1$, $\frac{a_1}{b_1} = \frac{2}{4\times3\times2}\frac{1}{\alpha^2}$. The smallest value for α^2 to ensure $a_1 \leq b_1$ is $1/12$. Assume $a_k/b_k \leq 1$ holds for some k and $\alpha^2 = 1/12$. Let us show that it holds for $k+1$. We have

$$\frac{a_{k+1}}{b_{k+1}} = \frac{a_k}{b_k}\frac{k+1}{(2k+3)(2k+4)}\frac{1}{\alpha^2} \leq \frac{k+1}{(2k+3)(2k+4)}\frac{1}{\alpha^2} \leq \frac{1}{2(2k+4)\alpha^2}.$$

One easily checks that for $k \geq 2$, the inequality $2(2k+4)\alpha^2 = (k+2)/3 > 1$ holds. We conclude that $a_k \leq b_k$ for all k. Hence (23) is component-wise larger than (22d). The claim is proved.

We can now bound the probability of interest:

$$\begin{aligned} \text{Prob}(Z \geq 0) &\leq \inf_{t>0} e^{tz_0}\prod_j E(e^{tz_j\xi_j}) && \text{(24a)} \\ &\leq \inf_{t>0} e^{tz_0}\prod_j e^{\frac{t^2}{12}z_j^2} && \text{(24b)} \\ &\leq e^{\inf_{t>0}(tz_0 + \frac{t^2}{12}\sum_j z_j^2)}. && \text{(24c)} \end{aligned}$$

Note that for $z_0 \geq 0$, the infimum in the right-hand side of (24c) is 1. In order to derive a useful bound, we shall assume $z_0 < 0$. Under this condition, the infimum is achieved at $t_{opt} = \frac{-6z_0}{||z||_2^2} > 0$ and

$$\text{Prob}(Z \geq 0) \leq e^{-\frac{3z_0^2}{||z||_2^2}}.$$

A sufficient condition to have the probability of constraint violation bounded above by ϵ is that the deterministic constraint, or robust equivalent,

$$z_0 + \sqrt{\frac{1}{3}\ln\frac{1}{\epsilon}}||z||_2 \leq 0 \tag{25}$$

be satisfied.

Finally, condition (25) can be made stronger without endangering the probabilistic statement. Indeed, let w be an arbitrary vector. Remembering that $-1 \leq \xi_j \leq 1$,

$$z_0 + z^T\xi = z_0 + (z-w)^T\xi + w^T\xi \leq z_0 + (z-w)^T\xi + ||w||_1. \tag{26}$$

Letting $\hat{z}_0 = z_0 + ||w||_1$ and $\hat{z} = z - w$, we have by (26)

$$\text{Prob}(z_0 + z^T\xi > 0) \leq \text{Prob}(\hat{z}_0 + \hat{z}^T\xi > 0).$$

We now apply (25) to the uncertain constraint $\hat{z}_0 + \hat{z}^T\xi \leq 0$ and obtain

$$\text{Prob}(\hat{z}_0 + \hat{z}^T\xi > 0) \leq \epsilon$$

whenever

$$z_0 + ||w||_1 + \sqrt{\frac{1}{3}\ln\frac{1}{\epsilon}}||z-w||_2 \leq 0$$

holds. This completes the first part of the proof.

To prove the last statement in the theorem, we just note that for $p = 1$, the coefficient $\sqrt{m}$ in the definition of κ_p comes into play because for any $a \in \mathbb{R}^m$ the ℓ_2-norm is bounded by $||a||_2 \leq \sqrt{m}\max_j |a_j| = \sqrt{m}||a||_\infty$.